MIDTERM: ALGEBRA I

Date: **9th September 2015**
The Total points is **110** and the maximum you can score is **100** points.

(1) (5+15=20 points) Let G be a set, $e \in G$ and $*$ be a binary operator on G. When is $(G, *, e)$ called a group? Let $f : \mathbb{N} \to \mathbb{Z}$ be the bijective function given by $f(m) = m/2$ if m is even and $f(m) = (1-m)/2$ if m is odd. For $a, b \in \mathbb{N}$, let $a * b = f^{-1}(f(a) + f(b))$. Show that $(\mathbb{N}, *, 1)$ is an abelian group.

(2) (5+5+10=20 points) Let G be a group and $x \in G$. Define the order of x. Let $x, y \in G$ be of finite order. Show that order of xy is finite if G is abelian. Show that this fails if G is not abelian.

(3) (10+5=15 points) Show that the group $\mathbb{R}/\mathbb{Z}$ is isomorphic to $(\{z \in \mathbb{C} : |z| = 1\}$. What is the image of $\mathbb{Q}/\mathbb{Z}$ under the isomorphism you found in the previous part?

(4) (5+10+5=20 points) Let G be a group. What is an automorphism of G? Show that the group of automorphisms of $(\mathbb{Z}/n, +)$ is isomorphic to $((\mathbb{Z}/n)^*, \cdot)$. Compute $Aut(\mathbb{Z})$.

(5) (5+5+10=20 points) When is a subgroup of G called a normal subgroup? Let D_{14} be the dihedral group of order 14. Show that the only nontrivial proper normal subgroup of D_{14} is the subgroup consisting of rotations. Let $\phi : D_{14} \to S_5$ be a group homomorphism. Show that the image $\mathrm{Im}(\phi)$ has atmost two elements.

(6) (5+10=15 points) Define the center of a group G. Suppose G has a unique element x of order 2. Then show that x is in the center of G.